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Degenerating Families of Meromorphic Functions on Compact Riemann Surfaces

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§0. Introduction. In this talk, I give a topological theory of degenerating families of meromorphic functions on compact Riemann surfaces. In [4], Matsumoto–Montesinos gave a topological theory of degenerating families of compact Riemann surfaces of genus ≥ 2. My theory can be regarded as an analogy to their theory in some sense. But my theory starts from branched coverings and is very explicit. Moreover they assumed that the total spaces are non-singular, while I don't. So the results thus obtained are slightly different.

§1. Terminology. A (non-constant) meromorphic function on a compact Riemann surface X of genus g is nothing but a surjective holomorphic mapping

$$f: X \longrightarrow \mathbb{P}^1 = \mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$$

of X onto the complex projective line $\mathbb{P}^1$. f is a surjective proper finite mapping. So, f can be regarded as a finite branched covering of $\mathbb{P}^1$. Put

$$R_f = \{ p \in X \mid f \text{ is not biholomorphic around } p \}, \quad B_f = f(R_f).$$

They are finite subsets of X and $\mathbb{P}^1$, respectively and are called the ramification locus and branch locus of f, respectively.

$$f : X - f^{-1}(B_f) \longrightarrow \mathbb{P}^1 - B_f$$

is an unramified covering. Its mapping degree is denoted by $\deg f$ and is called the degree of f.

In general, for a given complex manifold M, a <u>finite branched covering of</u> M is a finite proper holomorphic mapping

$$f : X \longrightarrow M$$

of an irreducible normal complex space X onto M. The <u>ramification locus</u> R_f, the <u>branch locus</u> B_f and the <u>degree of</u> f, $\deg f$, are defined as above. R_f and B_f are hypersurfaces of X and M, respectively. For a hypersurface B of M, f is said to <u>branch at most at</u> B if B_f is contained in B.

<u>Definition 1.</u> Branched coverings $f : X \longrightarrow M$ and $f' : X' \to M$ are <u>isomorphic</u> ($f \simeq f'$) if

$$\begin{tikzcd} X \arrow[rr, "\exists \psi"] \arrow[dr, "f"'] & \arrow[d, phantom, "\circlearrowleft"] & X' \arrow[dl, "f'"] \\ & M & \end{tikzcd}$$

, where ψ is a biholomorphic mapping. ⌟

Definition 2. Branched coverings $f: X \longrightarrow M$ and $f': X' \longrightarrow M'$ are equivalent ($f \sim f'$) (resp. topologically equivalent ($f \overset{\text{Top.}}{\sim} f'$)) if

$$\begin{array}{ccc} X & \xrightarrow{\exists \psi} & X' \\ f \downarrow & \circlearrowright & \downarrow f' \\ M & \xrightarrow{\exists \varphi} & M' \end{array}$$

, where ψ and φ are biholomorphic mappings (resp. orientation preserving homeomorphisms). ⌋

A theorem of Grauert–Remmert [2] asserts

Theorem 1 (Grauert–Remmert). Let B be a hypersurface of a complex manifold M and $f': X' \longrightarrow M - B$ be a finite unramified covering. Then there exists a unique (up to isomorphisms) finite covering $f: X \longrightarrow M$ which branches at most at B and is an extension of f'. ⌋

This theorem asserts that the correspondence $f' \longleftrightarrow f$ gives a categorical equivalence between unbranched coverings of $M - B$ and coverings of M branching at most at B. So, we can apply some terminology of unbranched coverings to the case of branched coverings. For example, covering transformations, Galois coverings, abelian coverings, cyclic coverings, etc..

Corollary. There is a one to one correspondence between $\{ f: X \longrightarrow M \mid f \text{ is a finite covering branching at most at } B \} / \simeq$ and $\{$ conjugacy classes of subgroups H of finite index of the fundamental group $\pi_1(M - B, q_0) \}$. ⌋

§ 2. Monodromy representations and checked patterns.

Let $f: X \longrightarrow M$ be a finite covering of a complex manifold M of degree d branching at most at a hypersurface B. The (permutation) monodromy representation

$$\Phi_f : \pi_1(M-B, q_0) \longrightarrow S_d \quad \text{(the } d\text{-th symmetric group)}$$

of $f: X - f^{-1}(B) \longrightarrow M - B$ is called the monodromy representation of f (see Fig. 1).

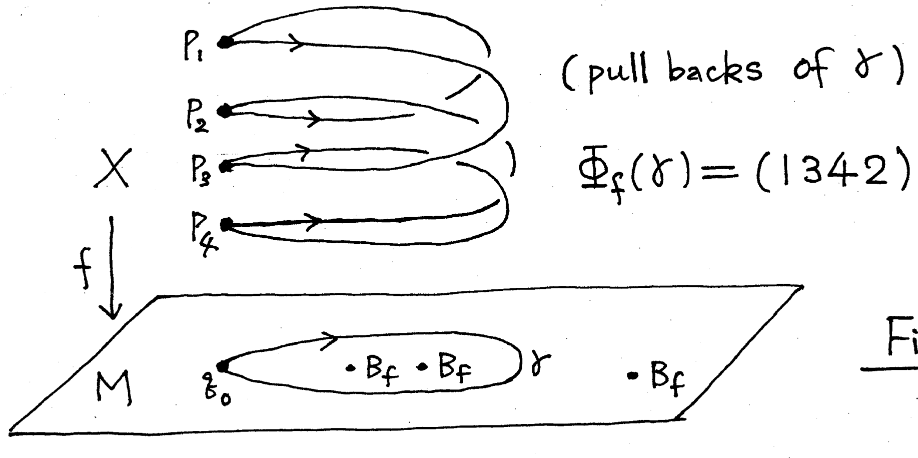


Fig. 1

$\operatorname{Im}\Phi_f$ is a transitive subgroup of S_d, for $X - f^{-1}(B)$ is connected.

By Theorem 1 and its corollary

Theorem 2. For a given homomorphism $\Phi : \pi_1(M-B, q_0) \longrightarrow S_d$ such that $\operatorname{Im}\Phi$ is transitive, there exists a unique (up to isomorphisms) covering $f: X \longrightarrow M$ of degree d branching at most at B such that $\Phi_f = \Phi$. ∎

Problem of Realization. Construct concretely the branched covering $f: X \longrightarrow M$ in the above theorem. ⌋

This is a difficult problem, which is studied in number theory in the case

$$M = \mathbb{P}^1 \text{ and } B = \{0, 1, \infty\} \quad \text{(see Schneps [5])}.$$

We consider the case $M = \mathbb{P}^1$ and construct $f: X \to \mathbb{P}^1$ topologically. The idea comes originally from Klein [3].

Let $B = \{g_1, g_2, \cdots, g_n\}$ be a set of n distinct points in $\mathbb{P}^1$. We draw a simple loop passing through every g_j oriented clockwisely as in Fig. 2.

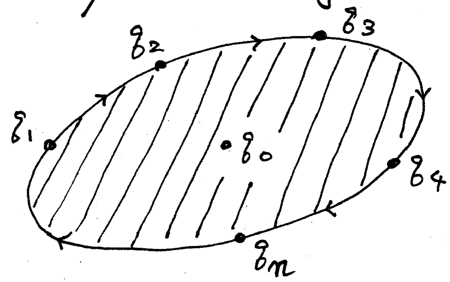


Fig. 2

We regard the inside area as a continent, which contains the reference point g_0, and the outside area as an ocean. We then pull back them over a covering $f: X \longrightarrow \mathbb{P}^1$ of degree d branching at most at B. Then we get a

checked pattern

of d continents and d oceans on X which is compatible with Φ_f.

Starting from $\Phi: \pi_1(\mathbb{P}^1 - B, g_0) \longrightarrow S_d$ such that $\mathrm{Im}\,\Phi$ is transitive, we do as follows: Put

$$M_j = \Phi(\gamma_j) \in S_d \quad \text{for } 1 \leq j \leq n,$$

where γ_j are lassos as in Fig. 3.

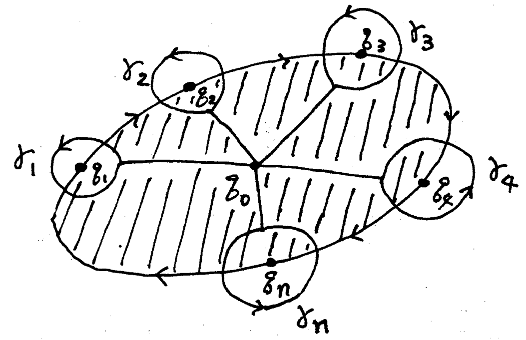


Fig. 3

Note that

$$\pi_1(\mathbb{P}^1 - B, q_0) = \langle \gamma_1, \cdots, \gamma_n \mid \gamma_n \cdots \gamma_1 = 1 \rangle,$$

$$M_n \cdots M_2 M_1 = 1 \in S_d .$$

Decompose each M_j into mutually prime cyclic permutations M_{jk} whose length is e_{jk}. Put (by Riemann-Hurwitz formula)

$$g = \frac{1}{2}\left\{ \Sigma_{j,k}(e_{jk} - 1) - 2d \right\} + 1 .$$

We prepare an oriented compact surface X of genus g. We then draw a checked pattern of d continents and d oceans on X which is compatible with Φ. This pattern describes topologically $f: X \longrightarrow \mathbb{P}^1$ such that $\Phi_f = \Phi$.

Example 1. $f: X \longrightarrow \mathbb{P}^1$, $(z, w) \longmapsto z$, where X is the Riemann surface of the algebraic function $w = w(z)$ given by the equation $w^3 - 3w - z = 0$. (The genus of X is 0.) Then $n = 3$, $d = 3$ and

$$M_1 = \Phi_f(\gamma_1) = (12), \; M_2 = \Phi_f(\gamma_2) = (13), \; M_3 = \Phi_f(\gamma_3) = (123).$$

See Fig. 4. The checked pattern in this case is as in Fig. 5.

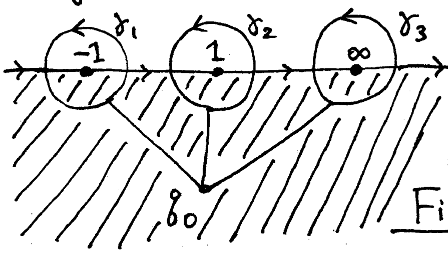


Fig. 4

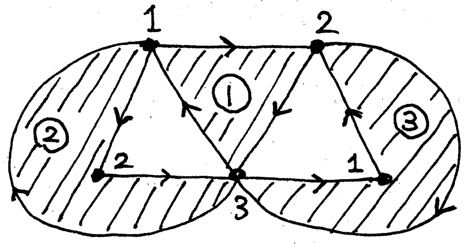


Fig. 5

(The points j in Fig. 5 are in $f^{-1}(g_j)$, while the number ⓘ in Fig. 5 denotes the i-th continent.)

Conversely we can read Φ_f from the checked pattern.

Example 2. $f: X \longrightarrow \mathbb{P}^1$, $(z, w) \longmapsto z$, where X is given by the equation $27zw^2(w-1)^2 - 4(w^2-w+1)^3 = 0$. This is a Galois covering with $\mathrm{Gal}(f) \simeq S_3$. (The genus of X is 0.) Then $n=3$, $d=6$ and

$$M_1 = \Phi_f(\gamma_1) = (154)(236), \quad M_2 = \Phi_f(\gamma_2) = (16)(24)(35),$$
$$M_3 = \Phi_f(\gamma_3) = (12)(34)(56).$$

See Fig. 6. The checked pattern in this case is as in Fig. 7.

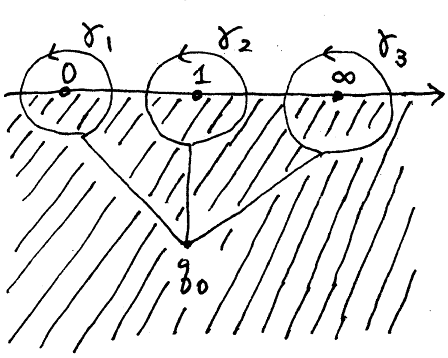


Fig. 6

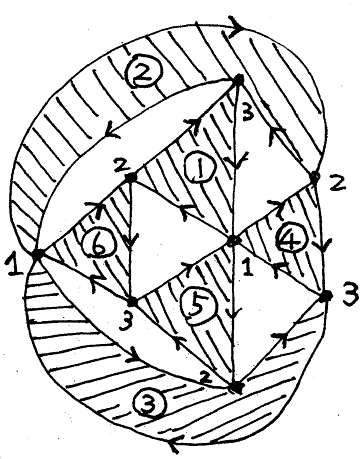


Fig. 7

Example 3. $f: X \longrightarrow \mathbb{P}^1$, $(z,w) \longmapsto z$, where X is given by the equation $w^3 - z^3 + 1 = 0$. (The genus of X is 1.) Then $n=3$, $d=3$ and

$$M_j = \Phi_f(\gamma_j) = (123) \quad \text{for } 1 \leq j \leq 3.$$

See Fig. 8. The checked patern in this case is in Fig. 9.

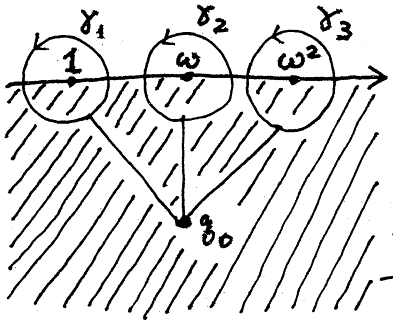


Fig. 8

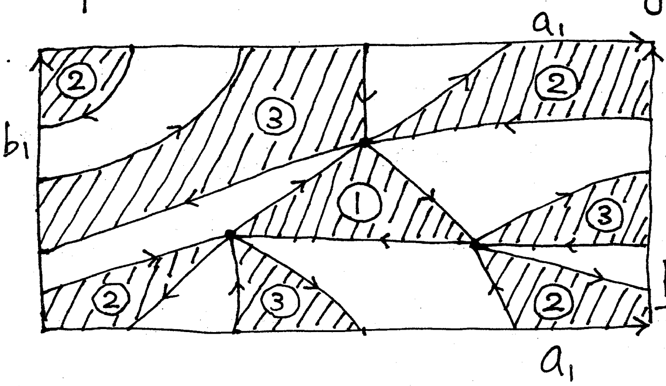


Fig. 9

Example 4. $f: X \longrightarrow \mathbb{P}^1$, $(z,w) \longmapsto z$, where X is given by the equation $w^3 - z^2(z-1)^2(z-2) = 0$. (The genus of X is 2.) Then $n=4$, $d=3$ and

$$M_1 = \Phi_f(\gamma_1) = (132), \quad M_2 = \Phi_f(\gamma_2) = (132),$$

$$M_3 = \Phi_f(\gamma_3) = (123), \quad M_4 = \Phi_f(\gamma_4) = (123).$$

See Fig 10. The checked pattern in this case is as in Fig. 11. (a_j and b_j for $1 \leq j \leq 2$ form a symplectic homology basis.)

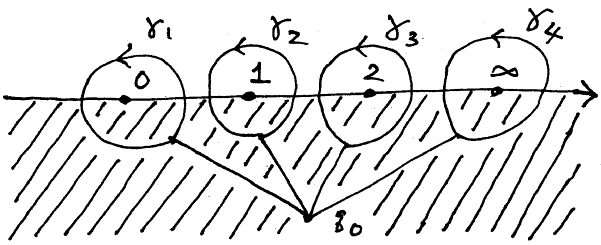


Fig. 10

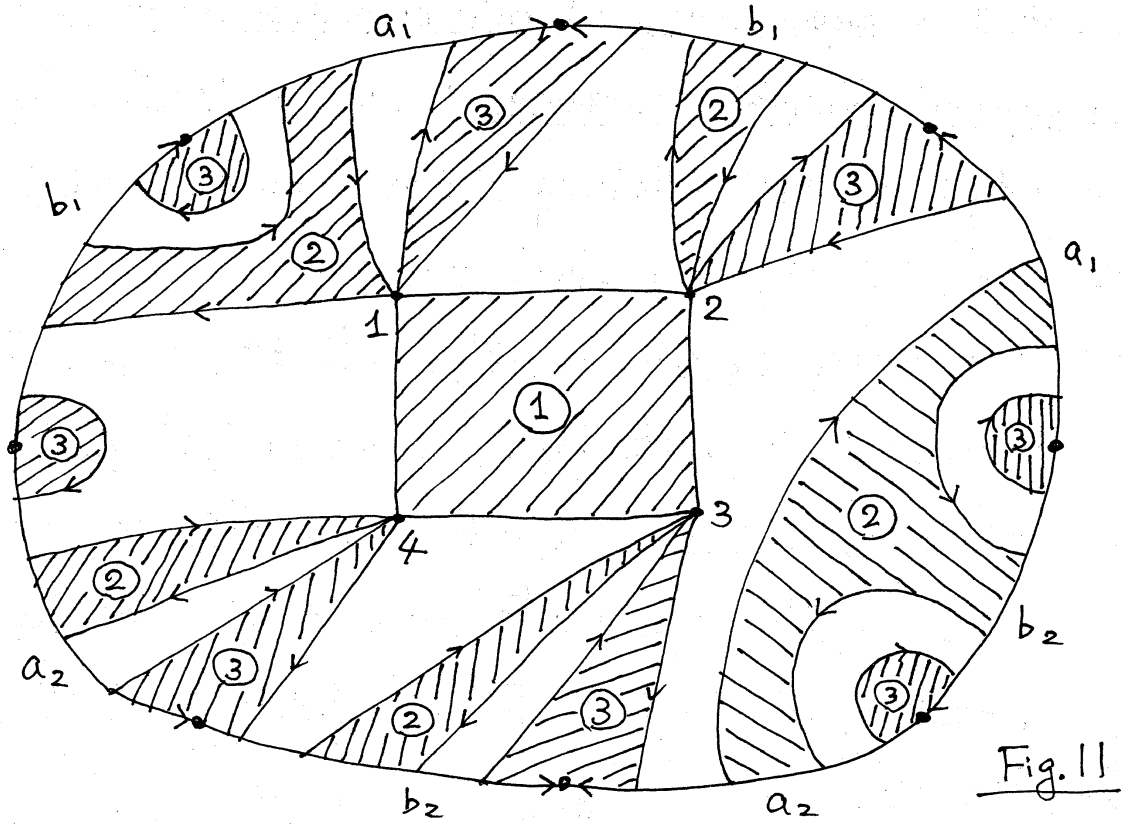


Fig. 11

So far, we gave examples in which the equations of coverings are given. But as I noted above, even if only the monodromies Φ are given and the equation of the coverings f are not given, then the topological picture (as Fig. 5 ~ Fig. 11) can be drawn.

§3. Degenerating families of meromorphic functions.

Let $\Delta = \Delta(0, \varepsilon) = \{t \in \mathbb{C} \mid |t| < \varepsilon\}$ be a disc and $\Delta^* = \Delta - \{0\}$ be the punctured disc. A finite branched covering

$$f : X \longrightarrow \Delta \times \mathbb{P}^1$$

is called a <u>degenerating family of meromorphic functions</u> (<u>of compact Riemann surfaces</u>) and is denoted by $f = \{f_t\}$ if the following 3 conditions are satisfied:

(1) $t \times \mathbb{P}^1 \not\subset B_f$ for every $t \in \Delta$.

(2) For every $t \in \Delta^*$, $t \times \mathbb{P}^1$ meets at n points transversally with B_f. (n is constant for $t \in \Delta^*$.)

(3) For every $t \in \Delta^*$, $f_t = f : X_t = f^{-1}(t \times \mathbb{P}^1) \longrightarrow t \times \mathbb{P}^1 = \mathbb{P}^1$ is a finite covering of degree $d = \deg f$ branching at $B_f \cap (t \times \mathbb{P}^1) = \{ q_1(t), \cdots, q_n(t) \}$.

See Fig. 12.

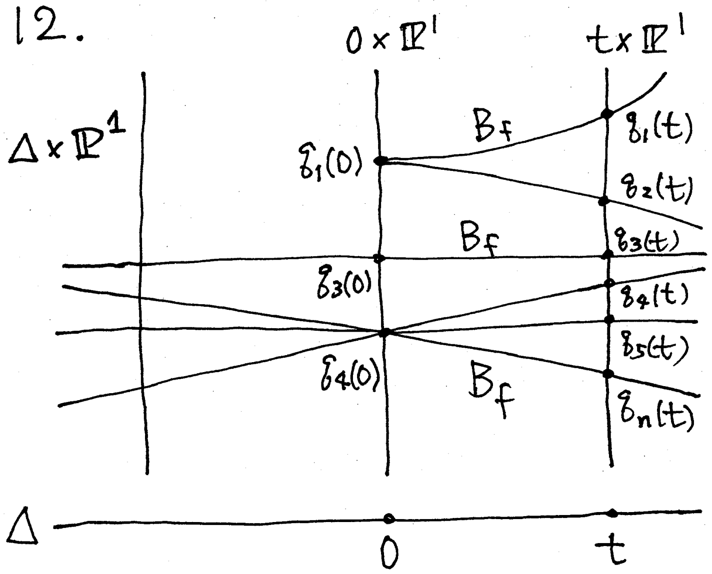


Fig. 12

The center fiber $X_0 = f^{-1}(0 \times \mathbb{P}^1)$ is a <u>degeneration of</u> X_t.

<u>Example 5.</u> $f = \{f_t\}$, $f_t : X_t \longrightarrow \mathbb{P}^1$, $(z, w) \mapsto z$, where $X_t : w^3 - 3tw - z = 0$. $M_1 = \Phi_t(\gamma_1) = (12)$, $M_2 = \Phi_t(\gamma_2) = (13)$, $M_3 = \Phi_t(\gamma_3) = (123)$. ($\Phi_t = \Phi_{f_t}$.)

See Fig. 13 and Fig. 14.

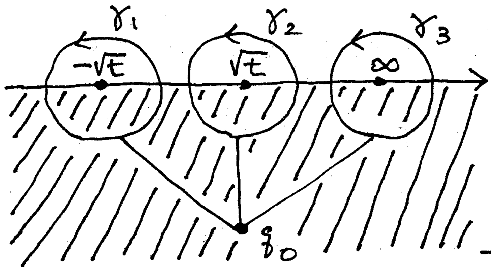


Fig. 13

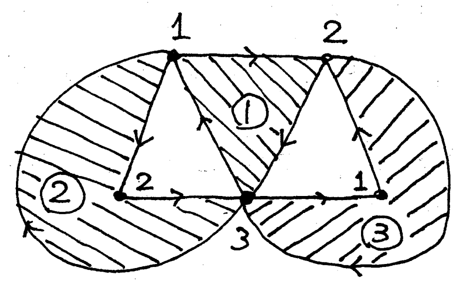


Fig. 14

As $t \longrightarrow 0$, the curves $1 \to 2$, whose initial points are 1 and end points are 2, converge to the point $1=2$, and we get the following picture of X_0:

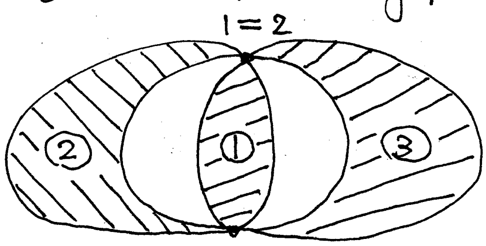


Fig. 15

In fact the equation $X_t: w^3 - 3tw - z = 0$ converges to $X_0: w^3 - z = 0$ as $t \longrightarrow 0$.

Example 6. $f = \{f_t\}$, $f_t: X_t \longrightarrow \mathbb{P}^1$, $(z, w) \longmapsto z$, where $X_t: w^2 - z(z-t)(z-1) = 0$. $M_j = \Phi_t(\gamma_j) = (1\,2)$ for $1 \leq j \leq 4$. See Fig. 16 and Fig. 17.

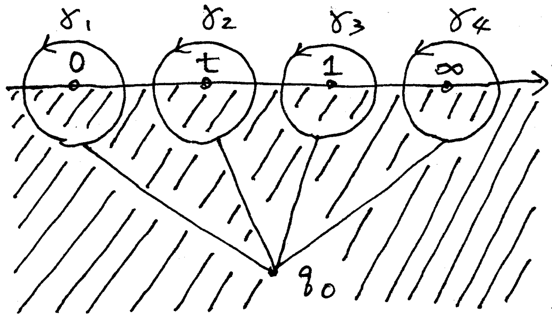


Fig. 16

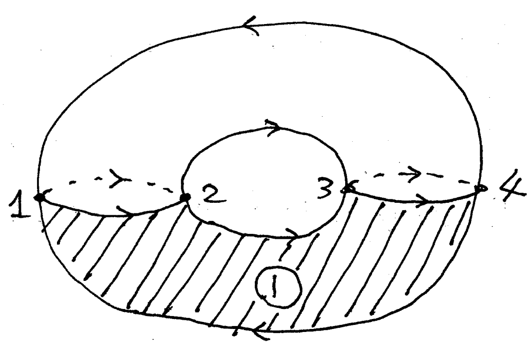
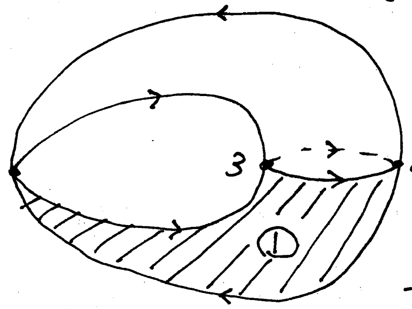


Fig. 17

(In Fig. 17, ② is the upper backside of the torus.)

As $t \longrightarrow 0$, the curves $1 \longrightarrow 2$ converges to the point $1 = 2$, and we get the following picture of X_0:



(② is the upper backside.)

Fig. 18

Example 7. $f = \{f_t\}$, $f_t : X_t \longrightarrow \mathbb{P}^1$, $(z, w) \longmapsto z$, where $X_t : w^2 - z^4 + t^4 = 0$. $M_j = \Phi_t(\gamma_j) = (12)$ for $1 \leq j \leq 4$. See Fig. 19 and Fig. 20.

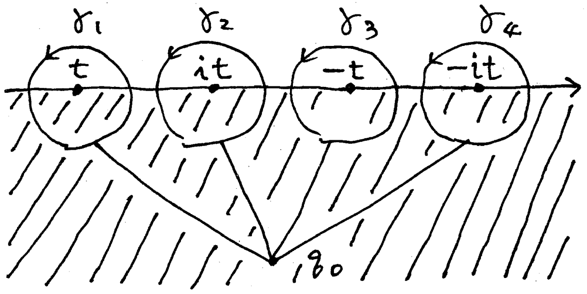


Fig. 19

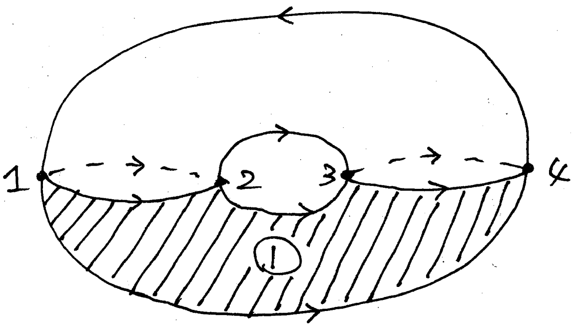


Fig. 20

(In Fig. 20, ② is the upper backside of the torus.)

As $t \longrightarrow 0$, the curves $1 \longrightarrow 2$, $2 \longrightarrow 3$, $3 \longrightarrow 4$ converge to the point $1 = 2 = 3 = 4$ and we get the following picture of X_0:

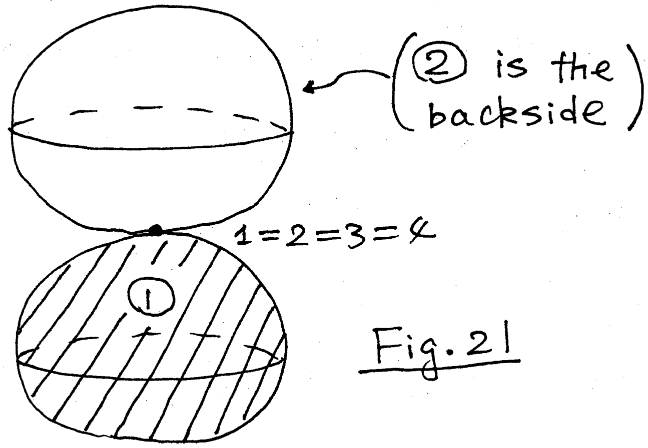


Fig. 21

Assertion. Topologically, the degenerating curve $X_0 = f^{-1}(0\times\mathbb{P}^1)$ can be described by $\Phi_t = \Phi_{f_t}$, where $t\in\Delta^*$ is a fixed point. ⌟

We explain this assertion as follows: For a family $\{f_t\}$, we assume for simplicity that

'$q_1(0) = \cdots = q_k(0)$ and other $q_j(0)$ $(k+1\le j\le n)$ are different from $q_1(0)$ and are mutually distinct.'

Put $M_1 = \Phi_t(\gamma_1), \cdots, M_n = \Phi_t(\gamma_n)$. Let H be the subgroup of S_d which is generated by $M_1, \cdots, M_k$. H may not be a transitive subgroup. We denote

$$\mathcal{O}_1, \cdots, \mathcal{O}_v$$

the orbits of H on $\{1, 2, \cdots, d\}$.

Definition 3. For a permutation $A\in S_d$, if A is written as $A = A_1\cdots A_w$, the product of mutually prime cyclic permutations, then we call the number $w = w(A)$ the <u>weight of</u> A. ($w(A)$ depends on d also. For example, if $d=4$ and $A=(123)$, then $w(A) = w((123)(4)) = 2$.) ⌋

Let $\chi(X_t)$ denote the Euler characteristic of X_t. Then we can easily show

<u>Theorem 3.</u> For $t \neq 0$,

(1) $\chi(X_t) = 2-2g = 2d - nd + \sum_{j=1}^{n} w(M_j)$,

(2) $\chi(X_0) = 2d - (n-k+1)d + v + \sum_{j=k+1}^{n} w(M_j)$,

(3) $\chi(X_0) - \chi(X_t) = d(k-1) + v - \sum_{j=1}^{k} w(M_j)$,

(4) $\chi(X_0) \geq \chi(X_t)$. ⌋

From Theorem 3 and som consideration, we get the following theorems:

<u>Theorem 4.</u>

(1) $f_0^{-1}(g_1(0))$ consists of $v+(n-k)$ points, among which v points can be identified with $\mathfrak{a}_1, \cdots, \mathfrak{a}_v$.

(2) Put $M_0 = M_k \cdots M_1$. Then M_0 induces a permutation $M_{0j} : \mathfrak{a}_j \longrightarrow \mathfrak{a}_j$. Under this notation, X_0 has $w(M_{0j})$ irreducible components at the point corresponding to $\mathfrak{a}_j$. ⌋

<u>Theorem 5.</u> The following 4 conditions are mutually

equivalent:

(1) X_0 is homeomorphic to X_t for $t \neq 0$.

(2) $\chi(X_0) = \chi(X_t)$ for $t \neq 0$.

(3) $d(k-1) = \sum_{j=1}^{k} w(A_j) - \nu$.

(4) $d(k-1) = \sum_{j=1}^{k} w(A_j) - w(A_0)$. $\lrcorner$

However the topological structure of $f = \{f_t\}$ is not determined by Φ_t alone. It depends also on the braid monodromy $\theta(\delta)$. Here

$$\delta : u \longmapsto t = t_0 e^{iu} \quad (0 \leq u \leq 2\pi)$$

is the loop around $t = 0$. ($t_0 \in \Delta^*$ is a fixed point.)

We assume that

'$g_j(t) \neq \infty$ for every $t \in \Delta$ and $1 \leq j \leq n$'.

Then $$\{ g_1(t_0 e^{iu}), \cdots, g_n(t_0 e^{iu}) \}_{0 \leq u \leq 2\pi}$$

gives an (Artin) braid of n strings, which is called the <u>braid monodromy of the curve B_f around</u> $t = 0$ and is denoted by $\theta(\delta)$.

The braid $\theta = \theta(\delta)$ can not be arbitrary. It is defined from a complex analytic curve B_f. So, such a braid we call a <u>complex analytic braid</u>.

Now, for a degenerating family $f = \{f_t\} : X \to \Delta \times \mathbb{P}^1$, we assume as above

$$(\Delta \times \{\infty\}) \cap B_f = \emptyset .$$

We fix a reference point $t_0 \in \Delta^*$ and put

$$q_j = q_j(t_0) \qquad \text{for } 1 \leq j \leq n.$$

Then the Artin braid group B_n naturally acts on the fundamental group $\pi_1(\mathbb{P}^1 - \{q_1, \cdots, q_n\}, q_0)$.

A theorem of Zariski–van Kampen (see e.g. Dimca [1]) asserts

<u>Theorem 6 (Zariski–van Kampen).</u> $\pi_1(\Delta \times \mathbb{P}^1 - B_f, q_0)$ $= \langle \gamma_1, \cdots, \gamma_n \mid \gamma_n \cdots \gamma_2 \gamma_1 = 1,\ \theta(\delta)\gamma_j = \gamma_j\ (1 \leq j \leq n) \rangle$, where γ_j are lassos as in Fig. 3 for $f_{t_0} : X_{t_0} \longrightarrow \mathbb{P}^1$. ∎

The monodromy representation Φ_f of $f : X \rightarrow \Delta \times \mathbb{P}^1$ is equal to $\Phi_{t_0} = \Phi_{f_{t_0}}$. By Theorem 6, Φ_{t_0} satisfies

$$\Phi_{t_0} \cdot \theta(\delta) = \Phi_{t_0} .$$

<u>Definition 4.</u> $f = \{f_t\}$ and $f' = \{f'_{t'}\}$ are said to be <u>topologically equivalent</u> if

$$\begin{tikzcd} X \arrow[r, "\exists \psi"] \arrow[d, "f"'] \arrow[dr, phantom, ""] & X' \arrow[d, "f'"] \\ \Delta \times \mathbb{P}^1 \arrow[r, "\exists \varphi"] \arrow[d] & \Delta' \times \mathbb{P}^1 \arrow[d] \\ \Delta \arrow[r, "\exists \eta"] & \Delta' \end{tikzcd}$$

where ψ, φ, η are orientation preserving homeomorphisms. ⌋

Using fundamental results in the theory of fiber bundles (see Steenrod [6]), we get the following theorem:

Theorem 7. There exists a one to one correspondence between {topological equivalence class of $f = \{f_t\}$, where f_{t_0} $(t_0 \neq 0)$ has the degree n and has prescribed n branch points} and $\{(\Phi, \theta) \mid \Phi$ is the representation class of $\Phi : \pi_1(\mathbb{P}^1 - \{q_1, \cdots, q_n\}, q_0) \longrightarrow S_d$ such that $\mathrm{Im}\,\Phi$ is transitive, $\theta \in B_n$ a complex analytic braid such that $\Phi \cdot \theta = \Phi\} / B_n$. ⌋

Here $\sigma \in B_n$ acts on (Φ, θ) as follows:

$$\sigma(\Phi, \theta) = (\Phi \cdot \sigma, \sigma^{-1} \theta \sigma).$$

Considering a trivial family, we get the following corollary which seems a known result.

Corollary. There exists a one to one correspondence between {topological equivalence class of $f : X \longrightarrow \mathbb{P}^1$ of degree d with prescribed n branch points} and $\{\Phi \mid \Phi$ is the representation class of $\Phi : \pi_1(\mathbb{P}^1 - \{q_1, \cdots, q_n\}, q_0) \longrightarrow S_d$ such that $\mathrm{Im}\,\Phi$ is transitive$\} / B_n$. ⌋

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